为叙述方便，首先证明如下的

引理 1 设 $P_i(\cos 2\theta_i, \sin 2\theta_i)$，$\theta_i \in (0, \frac{\pi}{2})$ 且 θ_i 各不相同，$i = 1,2,3,4$．直线 P_1P_2 与 P_3P_4 的交点为 $A(x_A, y_A)$，则

$$\begin{cases} x_A = \dfrac{\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} \\ y_A = \dfrac{\cos(\theta_1-\theta_2)\cos(\theta_3+\theta_4)-\cos(\theta_1+\theta_2)\cos(\theta_3-\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} \end{cases}.$$

证明 以下仅对直线 P_1P_2 与 P_3P_4 斜率存在的情形给出证明，斜率不存在时，容易验证结论成立.

P_1P_2 的斜率为：$\dfrac{\sin 2\theta_2 - \sin 2\theta_1}{\cos 2\theta_2 - \cos 2\theta_1} = -\dfrac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)}$，直线 P_1P_2 的方程为：$y - \sin 2\theta_1 = -\dfrac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)}(x - \cos 2\theta_2)$，整理得

$$y = -\frac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)}x + \frac{\cos(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \quad \cdots\cdots\cdots\cdots ①$$

同理，直线 P_3P_4 的方程为

$$y = -\frac{c\quad \theta_3 + o\theta}{s\quad \theta_3 + i\theta}x + \frac{s_4\quad \theta\ (-\theta}{n_4\quad \theta\ (+\theta} \quad \cdots\cdots\cdots\cdots ②$$

由①②消去 y 得，$x_A = \dfrac{\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)}$，将 x_A 代入①得

$$y = -\frac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)} \cdot \frac{\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} + \frac{\cos(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)}$$

$$= \frac{1}{\sin(\theta_1+\theta_2)}$$

$$\cdot[\cos(\theta_1-\theta_2) - \frac{\cos(\theta_1+\theta_2)(\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4))}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)}] \cdots\cdots\cdots ③$$

通分，并注意到 $\cos(\theta_1-\theta_2)\cdot\sin(\theta_1+\theta_2-\theta_3-\theta_4) = \cos(\theta_1-\theta_2)\sin((\theta_1+\theta_2)-(\theta_3+\theta_4))$

$= \cos(\theta_1-\theta_2)\sin(\theta_1+\theta_2)\cos(\theta_3+\theta_4) - \cos(\theta_1-\theta_2)\cos(\theta_1+\theta_2)\sin(\theta_3+\theta_4)$，所以③可化

为 $y=\dfrac{\cos(\theta_1-\theta_2)\cos(\theta_3+\theta_4)-\cos(\theta_1+\theta_2)\sin(\theta_3-\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)}$，此即 y_A，引理 1 证毕.

引理 2 设平面直角坐标系中三点 $A(x_A,y_A),P(x_P,y_P),D(x_D,y_D)$ 的坐标为依次为

$$\begin{cases} x_A=\dfrac{\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} \\ y_A=\dfrac{\cos(\theta_1-\theta_2)\cos(\theta_3+\theta_4)-\cos(\theta_1+\theta_2)\cos(\theta_3-\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} \end{cases},\quad \begin{cases} x_P=\dfrac{\cos(\theta_1+\theta_3)}{\cos(\theta_1-\theta_3)} \\ y_P=\dfrac{\sin(\theta_1+\theta_3)}{\cos(\theta_1-\theta_3)} \end{cases},$$

$$\begin{cases} x_D=\dfrac{\sin(\theta_2+\theta_3)\cos(\theta_4-\theta_1)-\cos(\theta_2-\theta_3)\sin(\theta_4+\theta_1)}{\sin(\theta_2+\theta_3-\theta_4-\theta_1)} \\ y_D=\dfrac{\cos(\theta_2-\theta_3)\cos(\theta_4+\theta_1)-\cos(\theta_2+\theta_3)\cos(\theta_4-\theta_1)}{\sin(\theta_2+\theta_3-\theta_4-\theta_1)} \end{cases}$$，则 A,P,D 三点共线.

证明 记 $a=\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4)-\cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)$，

$a'=\cos(\theta_1-\theta_2)\cos(\theta_3+\theta_4)-\cos(\theta_1+\theta_2)\cos(\theta_3-\theta_4)$，

$d=\sin(\theta_2+\theta_3)\cos(\theta_4-\theta_1)-\cos(\theta_2-\theta_3)\sin(\theta_4+\theta_1)$，

$d'=\cos(\theta_2-\theta_3)\cos(\theta_4+\theta_1)-\cos(\theta_2+\theta_3)\cos(\theta_4-\theta_1)$.

$p=\cos(\theta_1+\theta_3)$，$p'=\sin(\theta_1+\theta_3)$，$s=\sin(\theta_1+\theta_2-\theta_3-\theta_4)$，$s'=\sin(\theta_2+\theta_3-\theta_4-\theta_1)$，

$c=\cos(\theta_1-\theta_3)$.

因为

$$a=\frac{1}{2}[\sin(\theta_1+\theta_2+\theta_3-\theta_4)+\sin(\theta_1+\theta_2-\theta_3+\theta_4)]-\frac{1}{2}[\sin(\theta_3+\theta_4+\theta_1-\theta_2)+\sin(\theta_3+\theta_4-\theta_1+\theta_2)]$$

$$d=\frac{1}{2}[\sin(\theta_2+\theta_3+\theta_4-\theta_1)+\sin(\theta_2+\theta_3-\theta_4+\theta_1)]-\frac{1}{2}[\sin(\theta_1+\theta_4+\theta_1-\theta_2)+\sin(\theta_3+\theta_4-\theta_1+\theta_2)]$$

，所以 $a+d=\sin(\theta_1+\theta_2+\theta_3-\theta_4)-\sin(\theta_3+\theta_4+\theta_1-\theta_2)=2\cos(\theta_1+\theta_3)\sin(\theta_2-\theta_4)$

$a+d+p=(2\sin(\theta_2-\theta_4)+1)\cos(\theta_1+\theta_3)=(2\sin(\theta_2-\theta_4)+1)p$，

同理 $a'+d'+p'=(2\cos(\theta_2-\theta_4)+1)\sin(\theta_1+\theta_3)=(2\sin(\theta_2-\theta_4)+1)p'$，

$s+s'+c=2\sin(\theta_2-\theta_4)\cos(\theta_1-\theta_3)+\cos(\theta_1-\theta_3)=(2\sin(\theta_2-\theta_4)+1)\cos(\theta_1-\theta_3)$

$=(2\sin(\theta_2-\theta_4)+1)c$.

因为

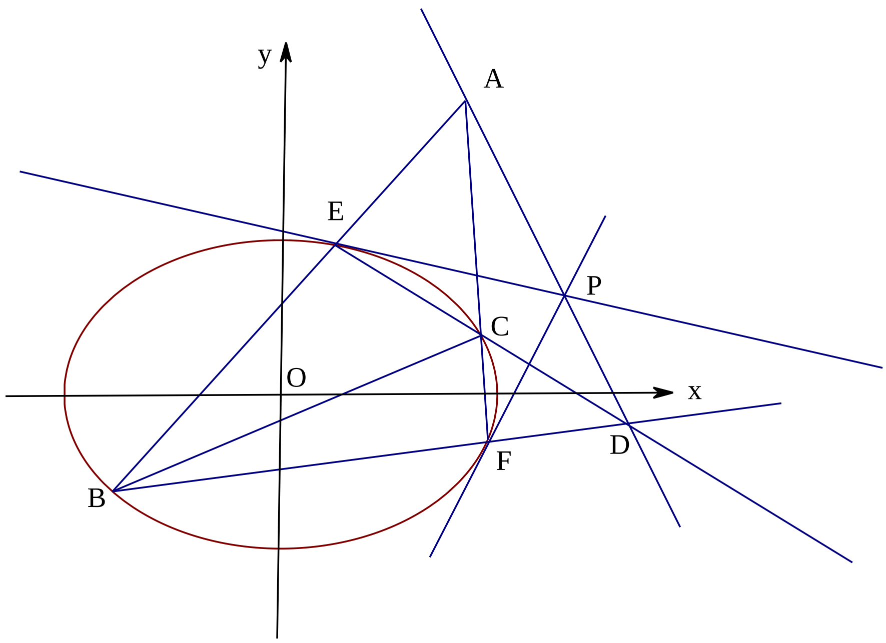
$$\Delta(A,D,P)=\begin{vmatrix} x_A & y_A & 1 \\ x_D & y_D & 1 \\ x_P & y_P & 1 \end{vmatrix}=\begin{vmatrix} \frac{a}{s} & \frac{a'}{s} & 1 \\ \frac{d}{s'} & \frac{d'}{s'} & 1 \\ \frac{p}{c} & \frac{p'}{c} & 1 \end{vmatrix}=$$

$$\frac{1}{ss'c}\begin{vmatrix} a & a' & s \\ d & d' & s' \\ p & p' & c \end{vmatrix}=\frac{1}{ss'c}\begin{vmatrix} a+d+p & a'+d'+p' & s+s'+c \\ d & d' & s' \\ p & p' & c \end{vmatrix}$$

$=\dfrac{2\sin(\theta_2-\theta_1)+1}{ss'c}\begin{vmatrix} p & p' & c \\ d & d' & s' \\ p & p' & c \end{vmatrix}=0$，所以 $A(x_A,y_A),P(x_P,y_P),D(x_D,y_D)$ 三点共线，引理 2 证毕.

以下利用引理 1 及引理 2 给原题的证明.

问题：如图，BC 是椭圆的弦，AB,AC 所在的直线与椭圆交于 E,F，过点 E,F 分别作椭圆的切线交于一点 P．直线 AP 与 BF 交于一点 D，则 D,C,E 三点共线.



证明　只要证明 A,P,D 三点共线.

设椭圆的方程为 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$（$a>b>0$）．取 $E(a\cos 2\theta_1, b\sin 2\theta_1), B(a\cos 2\theta_2, b\sin 2\theta_2)$,

$F(a\cos 2\theta_3, b\sin 2\theta_3), C(a\cos 2\theta_4, b\sin 2\theta_4)$，则直线 EB 的斜率为

$\dfrac{b\sin 2\theta_2 - b\sin 2\theta_1}{a\cos 2\theta_2 - a\cos 2\theta_1} = \dfrac{b}{a}\cdot\dfrac{2\cos(\theta_2+\theta_1)\sin(\theta_2-\theta_1)}{-2\sin(\theta_2+\theta_1)\sin(\theta_2-\theta_1)} = -\dfrac{b}{a}\cdot\dfrac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)}$，直线 EB 的方程为

$$y - b\sin 2\theta_1 = -\frac{b}{a}\cdot\frac{\cos(\theta_2+\theta_1)}{\sin(\theta_2+\theta_1)}(x - a\cos 2\theta_1)$$

，即

$$y = -\frac{b}{a}\cdot\frac{\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)}x + b\cdot\frac{\cos(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \quad \cdots\cdots\cdots\cdots ④$$

同理，直线 FC 的方程为

$$y = -\frac{b}{a}\cdot\frac{\cos(\theta_3+\theta_4)}{\sin(\theta_3+\theta_4)}x + b\cdot\frac{\cos(\theta_3-\theta_4)}{\sin(\theta_3+\theta_4)} \quad \cdots\cdots\cdots\cdots ⑤$$

由④⑤得 BE, FC 的交点 $A(x_A', y_B')$ 的坐标可以用引理 2 中的 x_A, y_A 表示为

$$\begin{cases} x_A' = a\cdot\dfrac{[\sin(\theta_1+\theta_2)\cos(\theta_3-\theta_4) - \cos(\theta_1-\theta_2)\sin(\theta_3+\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} = ax_A \\ y_A' = b\cdot\dfrac{\cos(\theta_1-\theta_2)\cos(\theta_3+\theta_4) - \cos(\theta_1+\theta_2)\cos(\theta_3-\theta_4)}{\sin(\theta_1+\theta_2-\theta_3-\theta_4)} = by_A \end{cases}$$

同理，直线 BF, EC 的交点 $D(x_D', y_D')$ 的坐标为

$$\begin{cases} x_D' = a\cdot\dfrac{\sin(\theta_2+\theta_3)\cos(\theta_4-\theta_1) - \cos(\theta_2-\theta_3)\sin(\theta_4+\theta_1)}{\sin(\theta_2+\theta_3-\theta_4-\theta_1)} = ax_D \\ y_D' = b\cdot\dfrac{\cos(\theta_2-\theta_3)\cos(\theta_4+\theta_1) - \cos(\theta_2+\theta_3)\cos(\theta_4-\theta_1)}{\sin(\theta_2+\theta_3-\theta_4-\theta_1)} = by_D \end{cases}$$

过 E, F 的两切线的交点为 $\begin{cases} x_P' = a\cdot\dfrac{\cos(\theta_1+\theta_3)}{\cos(\theta_1-\theta_3)} = ax_P \\ y_P' = b\cdot\dfrac{\sin(\theta_1+\theta_3)}{\cos(\theta_1-\theta_3)} = by_P \end{cases}$，由引理 2 的知，$\begin{vmatrix} x_A & y_A & 1 \\ x_D & y_D & 1 \\ x_P & y_P & 1 \end{vmatrix} = 0$，

所以，$\Delta(A,D,P) = \begin{vmatrix} x_A' & y_A' & 1 \\ x_D' & y_D' & 1 \\ x_P' & y_P' & 1 \end{vmatrix} = \begin{vmatrix} ax_A & by_A & 1 \\ ax_D & by_D & 1 \\ ax_P & by_P & 1 \end{vmatrix} = ab\begin{vmatrix} x_A & y_A & 1 \\ x_D & y_D & 1 \\ x_P & y_P & 1 \end{vmatrix} = 0$，即 A, P, D 共线.