已知 $f(x)=\int_1^x \frac{\sin(xt)}{t}\,\mathrm{d}\,t$，求 $\int_0^1 xf(x)\,\mathrm{d}\,x$。

解 我们先研究一下 $f'(x)$ 如何。考虑 $x>0$ 时，取充分小的 $|\Delta x|\neq 0$ 使得 $x+\Delta x>0$，我们有

$$
\begin{aligned}
& f(x+\Delta x)-f(x)\\
={}& \int_1^{x+\Delta x}\frac{\sin((x+\Delta x)t)}{t}\,\mathrm{d}\,t-\int_1^x\frac{\sin(xt)}{t}\,\mathrm{d}\,t\\
={}& \int_1^{x+\Delta x}\frac{\sin(xt)\cos(\Delta xt)+\cos(xt)\sin(\Delta xt)}{t}\,\mathrm{d}\,t-\int_1^x\frac{\sin(xt)}{t}\,\mathrm{d}\,t\\
={}& \int_1^{x+\Delta x}\frac{\sin(xt)+\sin(xt)(\cos(\Delta xt)-1)+\cos(xt)\sin(\Delta xt)}{t}\,\mathrm{d}\,t-\int_1^x\frac{\sin(xt)}{t}\,\mathrm{d}\,t\\
={}& \int_1^{x+\Delta x}\frac{\sin(xt)}{t}\,\mathrm{d}\,t-\int_1^x\frac{\sin(xt)}{t}\,\mathrm{d}\,t+\int_1^{x+\Delta x}\frac{-2\sin(xt)\sin^2(\Delta xt/2)+\cos(xt)\sin(\Delta xt)}{t}\,\mathrm{d}\,t\\
={}& \int_x^{x+\Delta x}\frac{\sin(xt)}{t}\,\mathrm{d}\,t-2\int_1^{x+\Delta x}\frac{\sin(xt)\sin^2(\Delta xt/2)}{t}\,\mathrm{d}\,t+\int_1^{x+\Delta x}\frac{\cos(xt)\sin(\Delta xt)}{t}\,\mathrm{d}\,t,
\end{aligned}
$$

由积分学中值定理，有

$$\int_x^{x+\Delta x}\frac{\sin(xt)}{t}\,\mathrm{d}\,t=\frac{\sin(x(x+\theta\cdot\Delta x))}{x+\theta\cdot\Delta x}\cdot\Delta x,$$

其中 $\theta\in[0,1]$，于是有

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\sin(x(x+\theta\cdot\Delta x))}{x+\theta\cdot\Delta x}-2\int_1^{x+\Delta x}\frac{\sin(xt)\sin^2(\Delta xt/2)}{\Delta xt}\,\mathrm{d}\,t+\int_1^{x+\Delta x}\frac{\cos(xt)\sin(\Delta xt)}{\Delta xt}\,\mathrm{d}\,t,$$

取极限，得到

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\sin(x^2)}{x}+\int_1^x\cos(xt)\,\mathrm{d}\,t=\frac{\sin(x^2)}{x}+\left.\frac{\sin(xt)}{x}\right|_1^x=\frac{2\sin(x^2)-\sin x}{x},$$

即当 $x>0$ 时有

$$f'(x)=\frac{2\sin(x^2)-\sin x}{x}.$$

时间关系，$x\leqslant 0$ 的就先不考虑了，估计也一样，现直接用 $x>0$ 的结果就够了。由分部积分法，有

$$
\begin{aligned}
\int_0^1 xf(x)\,\mathrm{d}\,x &= \frac{1}{2}\int_0^1 f(x)(x^2)'\,\mathrm{d}\,x\\
&= \frac{1}{2}\,f(x)x^2\Big|_0^1-\frac{1}{2}\int_0^1 x^2f'(x)\,\mathrm{d}\,x\\
&= \frac{1}{2}\,f(x)x^2\Big|_0^1-\frac{1}{2}\int_0^1 x(2\sin(x^2)-\sin x)\,\mathrm{d}\,x\\
&= \frac{1}{2}f(1)-\frac{1}{2}\int_0^1\sin(x^2)\,\mathrm{d}\,x^2+\frac{1}{2}\int_0^1 x\sin x\,\mathrm{d}\,x\\
&= -\frac{1}{2}\int_0^1\sin y\,\mathrm{d}\,y-\frac{1}{2}\int_0^1 x(\cos x)'\,\mathrm{d}\,x\\
&= \frac{1}{2}\,\cos y\Big|_0^1-\frac{1}{2}\,x\cos x\Big|_0^1+\frac{1}{2}\int_0^1\cos x\,\mathrm{d}\,x\\
&= \frac{\cos 1-1}{2}-\frac{\cos 1}{2}+\frac{\sin 1}{2}\\
&= \frac{\sin 1-1}{2}.
\end{aligned}
$$

累啊！ □