From Lalescu's sequence to a Gamma function limit

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Abstract

In this article we study a generalisation of Lalescu's sequence involving the Gamma function.

Introduction

Traian Lalescu, a great Romanian mathematician, has proposed in *Gazeta Matematica* [4] the study of the sequence $(L_n)_{n\in N}$ with the general term:

$$L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}.$$

The sequence $(L_n)_{n\in N}$, also known as Lalescu's sequence, has been studied by many Romanian mathematicians and it has been shown that it converges to $1/e$. For a recent study of this sequence as well as related Lalescu-like sequences, we recommend [3] and [5]. In this article we consider the following generalisation of $(L_n)_{n\in N}$ involving the Gamma function.

Theorem 1. *Let $m \geq 0$ be a natural number and let f be a polynomial of degree m whose coefficient in the leading term is positive. Then,*

(a) $$\lim_{x\to\infty}((f(x+2)\Gamma(x+2))^{1/(1+x)} - (f(x+1)\Gamma(x+1))^{1/x}) = \frac{1}{e}.$$

(b) $$\lim_{x\to\infty} x\left(((f(x+2)\Gamma(x+2))^{1/(1+x)} - (f(x+1)\Gamma(x+1))^{1/x} - \frac{1}{e}\right) = \frac{1}{e}\left(m+\frac{1}{2}\right).$$

Before proving the theorem, we collect some known results about the Gamma function. We need, in our analysis, the following limit

$$\lim_{x\to\infty}\frac{(\Gamma(x+1))^{1/x}}{x} = \frac{1}{e}, \tag{1}$$

which can be proved by an application of Stirling's formula.

The psi function, also known as digamma function, is defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$, and its derivative verifies the following asymptotic expansion, [1]:

$$\psi'(z) \approx \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \cdots, \qquad z\to\infty, \quad |\arg z| < \pi. \tag{2}$$

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It follows, based on (2), that

$$\lim_{x\to\infty} x(x\psi'(x+1)-1) = -\frac{1}{2}\,. \tag{3}$$

Anderson [2] proved that if $h(x) = x\psi(x+1) - \ln\Gamma(x+1)$, then

$$\frac{h(x)}{x} = 1 + O\left(\frac{\ln x}{x}\right). \tag{4}$$

Now we are ready to prove Theorem 1. To make the calculations easier to follow we will write $\sqrt[x]{f(x)}$ instead of $(f(x))^{1/x}$.

Proof of Theorem 1

Proof. (a) Let f be a polynomial of degree m whose coefficient in the leading term is a positive real number. It follows that f is positive for large values of x and $(f(x))^{1/x}$ is well defined. We will prove that

$$\lim_{x\to\infty} (f(x+2)\Gamma(x+2))^{1/(1+x)} - (f(x+1)\Gamma(x+1))^{1/x} = \frac{1}{e}\,. \tag{5}$$

Let $u(x) = \sqrt[x]{\Gamma(x+1)f(x+1)}$ and $V(x) = u(x+1) - u(x)$. The mean value theorem implies that $V(x) = u'(c)$ for some $c \in (x, x+1)$. Thus, to prove (5) it suffices to show that $\lim_{x\to\infty} u'(x) = 1/e$. A straightforward calculation shows that,

$$\begin{aligned} u'(x) = {} & \frac{\sqrt[x]{\Gamma(x+1)}}{x}\sqrt[x]{f(x+1)} \\ & \times \left(\frac{\psi(x+1)x - \ln\Gamma(x+1)}{x} + \frac{x(f'(x+1)/f(x+1)) - \ln f(x+1)}{x}\right). \end{aligned}$$

On the other hand, since f is a polynomial, we obtain that

$$\lim_{x\to\infty} \sqrt[x]{f(x+1)} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{x(f'(x+1)/f(x+1)) - \ln f(x+1)}{x} = 0\,. \tag{6}$$

Combining (1), (4) and (6) we get that $\lim_{x\to\infty} u'(x) = 1/e$.

(b) Let $L = \lim_{x\to\infty} x((f(x+2)\Gamma(x+2))^{1/(1+x)} - (f(x+1)\Gamma(x+1))^{1/x} - 1/e)$. An application of l'Hopital's rule shows that

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{u(x+1) - u(x) - 1/e}{1/x} \\ &= \lim_{x\to\infty} \frac{u'(x+1) - u'(x)}{-1/x^2} \\ &= -\lim_{x\to\infty} x^2(u'(x+1) - u'(x)). \end{aligned}$$

Using the mean value theorem we obtain that $u'(x+1) - u'(x) = u''(c)$ for some $c \in (x, x+1)$, so $L = -\lim_{x\to\infty} x^2 u''(x)$. We note that,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[x]{\Gamma(x+1)} &= \sqrt[x]{\Gamma(x+1)}\ \frac{\psi(x+1)x - \ln\Gamma(x+1)}{x^2}\\ &= \sqrt[x]{\Gamma(x+1)}\ a(x)\\ &= \sqrt[x]{\Gamma(x+1)}\ \frac{h(x)}{x^2},\end{aligned}$$

where

$$a(x) = \frac{\psi(x+1)x - \ln\Gamma(x+1)}{x^2} \quad \text{and} \quad h(x) = \psi(x+1)x - \ln\Gamma(x+1).$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ \sqrt[x]{f(x+1)} &= \sqrt[x]{f(x+1)}\ \frac{(f'(x+1)/f(x+1))x - \ln f(x+1)}{x^2}\\ &= \sqrt[x]{f(x+1)}\ b(x)\\ &= \sqrt[x]{f(x+1)}\ \frac{g(x)}{x^2},\end{aligned}$$

where

$$b(x) = \frac{(f'(x+1)/f(x+1))x - \ln f(x+1)}{x^2}$$

and

$$g(x) = \frac{f'(x+1)}{f(x+1)}\,x - \ln f(x+1)\,.$$

A calculation shows that,

$$a'(x) = \frac{\psi'(x+1)}{x} - \frac{2a(x)}{x} = \frac{\psi'(x+1)}{x} - \frac{2h(x)}{x^3}$$

and

$$b'(x) = \frac{(f'(x+1)/f(x+1))'}{x} - \frac{2g(x)}{x^3}.$$

It follows, based on formula $(fg)'' = f''g + 2f'g' + fg''$, that

$$\begin{aligned}u''(x) &= \sqrt[x]{\Gamma(x+1)}\,\sqrt[x]{f(x+1)}((a(x)+b(x))^2 + a'(x) + b'(x))\\ &= \sqrt[x]{\Gamma(x+1)}\,\sqrt[x]{f(x+1)}\\ &\quad\times\left(\frac{(h(x)+g(x))^2}{x^4} + \frac{\psi'(x+1)}{x} - \frac{2h(x)}{x^3}\right.\\ &\qquad\left. + \frac{(f'(x+1)/f(x+1))'}{x} - \frac{2g(x)}{x^3}\right).\end{aligned}$$

Thus, $x^2u''(x)$ is equal to

$$\begin{aligned}&\frac{\sqrt[x]{\Gamma(x+1)}}{x}\,\sqrt[x]{f(x+1)}\\ &\quad\times\left(\frac{(h(x)+g(x))^2}{x} + x^2\psi'(x+1) - 2h(x) - 2g(x) + x^2\left(\frac{f'(x+1)}{f(x+1)}\right)'\right).\end{aligned}$$

Therefore we obtain that,

$$L = -\frac{1}{e}\lim_{x\to\infty}\left(\frac{(h(x)+g(x))^2}{x} + x^2\psi'(x+1) - 2h(x) - 2g(x) + x^2\left(\frac{f'(x+1)}{f(x+1)}\right)'\right). \tag{7}$$

On the other hand, a calculation shows that,

$$\lim_{x\to\infty} x^2\left(\frac{f'(x+1)}{f(x+1)}\right)' = -m. \tag{8}$$

Combining (3), (7) and (8) we obtain that

$$L = \frac{1}{e}\left(\frac{1}{2}+m\right) - \frac{l}{e},$$

where

$$\begin{aligned} l &= \lim_{x\to\infty}\left(\frac{(h(x)+g(x))^2}{x} - 2h(x) - 2g(x) + x\right)\\ &= \lim_{x\to\infty} x\left(\frac{h^2(x)+2h(x)g(x)+g^2(x)}{x^2} - \frac{2h(x)}{x} - \frac{2g(x)}{x} + 1\right). \end{aligned}$$

Using (4) we obtain that

$$\begin{aligned} l &= \lim_{x\to\infty} x\left(\left(\frac{h(x)}{x}-1\right)^2 + \frac{2g(x)}{x}\left(\frac{h(x)}{x}-1\right) + \frac{g^2(x)}{x^2}\right)\\ &= \lim_{x\to\infty}\left(xO^2\left(\frac{\ln x}{x}\right) + 2g(x)O\left(\frac{\ln x}{x}\right) + \frac{g^2(x)}{x}\right) = 0. \end{aligned}$$

The last equality is justified by the following calculations

$$xO^2\left(\frac{\ln x}{x}\right) = \frac{\ln^2 x}{x}\left[\frac{O(\ln x/x)}{\ln x/x}\right]^2 \to 0,$$

$$2g(x)O\left(\frac{\ln x}{x}\right) = 2\frac{O(\ln x/x)}{\ln x/x}\left(\frac{xf'(x+1)}{f(x+1)}\frac{\ln x}{x} - \ln f(x+1)\frac{\ln x}{x}\right) \to 0,$$

and

$$\frac{g^2(x)}{x} = \frac{1}{x}\left(x\frac{f'(x+1)}{f(x+1)}\right)^2 - 2x\frac{f'(x+1)}{f(x+1)}\frac{\ln f(x+1)}{x} + \frac{\ln^2 f(x+1)}{x} \to 0.$$

Therefore we obtain that $L = \frac{1}{e}(m+\frac{1}{2})$ and the theorem is proved.

Corollary 2. *Let f and g be polynomials of degree m and n whose leading coefficients are positive real numbers. Then,*

$$\lim_{x\to\infty} \frac{(f(x+2)\Gamma(x+2))^{1/(1+x)} - (f(x+1)\Gamma(x+1))^{1/x} - 1/e}{(g(x+2)\Gamma(x+2))^{1/(1+x)} - (g(x+1)\Gamma(x+1))^{1/x} - 1/e} = \frac{2m+1}{2n+1}.$$

Corollary 3. *Let f and g be polynomials of degree m and n whose leading coefficients are positive real numbers and let m, n, p and q be nonnegative integers such that $p \neq q$. Then,*

$$\lim_{x\to\infty} \frac{\begin{array}{l}(f(x+m+1)\Gamma(x+m+1))^{1/(m+x)} \\ \qquad - (f(x+n+1)\Gamma(x+n+1))^{1/(x+n)} - ((m-n)/e)\end{array}}{\begin{array}{l}(g(x+p+1)\Gamma(x+p+1))^{1/(p+x)} \\ \qquad - (g(x+q+1)\Gamma(x+q+1))^{1/(x+q)} - ((p-q)/e)\end{array}}$$
$$= \frac{(2m+1)(m-n)}{(2n+1)(p-q)}.$$

Proof. Let

$$a_m(x) = (f(x+m+1)\Gamma(x+m+1))^{1/(m+x)}$$

and

$$b_p(x) = (g(x+p+1)\Gamma(x+p+1))^{1/(p+x)}.$$

We have that,

$$\begin{aligned}&\frac{a_m(x) - a_n(x) - (m - n/e)}{b_p(x) - b_q(x) - (p - q/e)} \\ &\quad = \frac{(a_m(x) - a_{m-1}(x) - (1/e)) + \cdots + (a_{n+1}(x) - a_n(x) - (1/e))}{(b_p(x) - b_{p-1}(x) - (1/e)) + \cdots + (b_{q+1}(x) - b_q(x) - (1/e))}.\end{aligned} \tag{9}$$

Multiplying by x both the numerator and the denominator of (9) and taking the limit completes the proof.

The following corollary gives a generalisation of Lalescu's sequence.

Corollary 4. *Let f be a polynomial of degree m with a positive leading coefficient. Then,*

(a) $$\lim_{n\to\infty} \left(\sqrt[n+1]{f(n+2)(n+1)!} - \sqrt[n]{f(n+1)n!} \right) = \frac{1}{e}.$$

(b) $$\lim_{n\to\infty} n \left(\sqrt[n+1]{f(n+2)(n+1)!} - \sqrt[n]{f(n+1)n!} - \frac{1}{e} \right) = \frac{1}{e}\left(m + \frac{1}{2}\right).$$

Remark 1. Letting $f = 1$ in the preceding corollary we obtain that the limit of Lalescu's sequence is $1/e$ and that the second term of the asymptotic expansion of L_n is $1/(2en)$, in other words

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e},$$

and

$$\lim_{n\to\infty} n\left(L_n - \frac{1}{e}\right) = \lim_{n\to\infty} n\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} - \frac{1}{e}\right) = \frac{1}{2e}.$$

A natural question is to determine the asymptotic expansion of L_n.

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